Measurement Errors: Connections and Solutions

Andrey Feuerverger
University of Toronto
andrey@fisher.utstat.toronto.edu
Jiayang Sun
Case Western Reserve University
jiayang@sun.cwru.edu

Workshop on Developments and Challenges in Mixture Models, Bump Hunting and Measurement Error Models, Cleveland, June 2-4, 2002

Outline

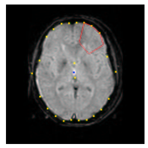
- Introduction
- Connections
- Challenges
- Naive Solutions
- Better Solutions
- Discussion and Remarks

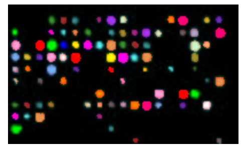
Introduction

Many interesting problems can be formulated as studies about data with measurement errors:

- Signal processing: output is the original signal coupled with the filter's impulse and errors

- (Microarray) Image analysis: observable is a blurred image





Ref: http://www.niac.man.ac.uk/projects/mccollum.html

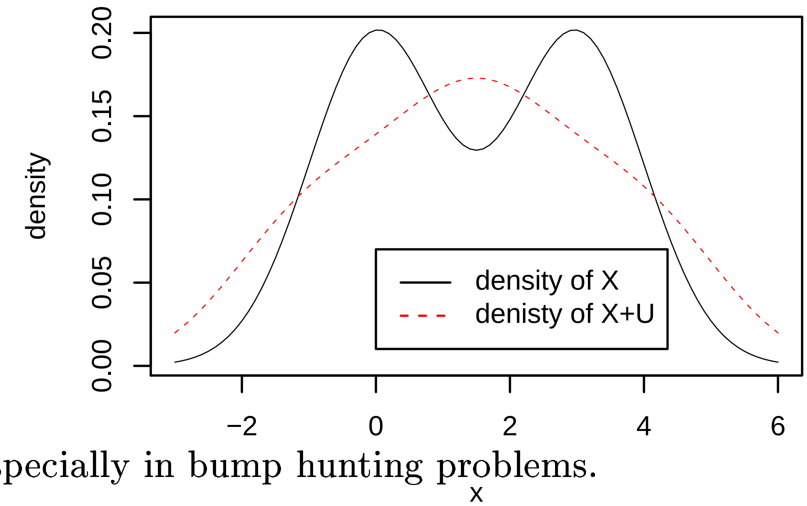
http://www.cmis.csiro.au/IAP/spot.htm

- Astronomy: data are often subject to measurement errors

observed velocity km/sec	id	fiber	peak	width	tdr	error
133.8519	318___15.	81	0.09	102.07	3.08	19.633
221.1162	337___15.	32	0.12	120.29	3.19	22.571
-120.3765	343___14.	88	0.15	129.35	4.35	18.988
144.1023	371___15.	84	0.20	114.98	5.82	13.233
-119.0284	202___14.	65	0.20	149.91	5.75	17.441

....

DO NOT ignore measurement errors



especially in bump hunting problems.

Road Map - connections

- $Y = m(X) + \epsilon$ Ruppert

  Observe Y and $W = X + U$. **What** is $m(x)$?

- $Y = X + U$

  Observe Y. **What** is density f_X of X?

- $Y(t) = K(x(t)) + U(t)$ related

  Observe Y at t and know k. **What** is $x(t)$?

  – inverse porblems such as signal/image analyses

Challenges

Observe: $Y_j = X_j + U_j$, independently,

where: $X_j \sim f_X$, $U_j \sim f_j$.

1. If $f_j = f_0$, homogeneous; otherwise nonhomogeneous, e.g. $f_j = U(-h_j, h_j)$ or $N(0, h_j)$.
2. If f_j is Uniform, it is a blessing and a challenge!

$\Longrightarrow$ implications for other error distributions.

Deconvolution

- Homogeneous case: $Y_i = X_i + U_i$, $X_i \overset{iid}{\sim} f_X, U_i \overset{iid}{\sim} f_U$

Density: $f_Y = f_X * F_U$

Characteristic function: $\varphi_Y = \varphi_X \cdot \varphi_U$

Naive estimate:

$$\hat{f}_X(x) = \frac{1}{2\pi} \int \frac{\hat{\varphi}_Y(t)}{\varphi_U(t)} e^{-itx} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\frac{1}{n}\sum_j e^{itY_j}}{\varphi_U(t)} \, e^{-itx} \, dt$$

Modified estimate:

$$\hat{f}_X(x) = \frac{1}{2\pi n} \int_{-\infty}^{\infty} \frac{\sum_j e^{itY_j}}{\varphi_U(t)} \, W(a_n t) \, e^{-itx} \, dt \,, \qquad (1)$$

where $W(t)$ is a 'window' function, and $a_n \downarrow 0$.

Note: If $W = \varphi_K$ where K is a kernel, then (1) is the kernel deconvolution estimate studied by many authors, Carrol and Hall(88), Fan(91), Zhang(90) ...

Problem: The convergence rate of $\hat{f}_X$ is extremely slow and gets worse as f_U gets smoother. The uniform distribution is neither ordinary nor super smooth as defined by Fan (1991). However, note that $\varphi_U(t) = \sin(ht)/(ht)$ if $U \sim U(-h, h)$, so,

$$\hat{f}_X(x) = \frac{1}{2\pi n} \int_{-\infty}^{\infty} \frac{\sum_j e^{itY_j}}{\sin(t)/t} \, W(a_n t) \, e^{-itx} \, dt \,,$$

where WOLG we took h=1. There are singularities at $k\pi$! The numerator is not zero when the denominator is though $E(\text{numerator}) = 0$ then.

Solutions

1. Abandon the characteristic functions. This opens a completely new horizon.

   Sun, Morrison, Harding and Woodroofe (2002)

2. Smooth out the singularity points. There are several possibilities, including ones using splines and Shanoon sampling formula. Some lead to optimal estimators!

Both work for the nonhomogeneous case and have implications for non-uniform errors.

Solution 1
Abandon Ch functions - SMHW

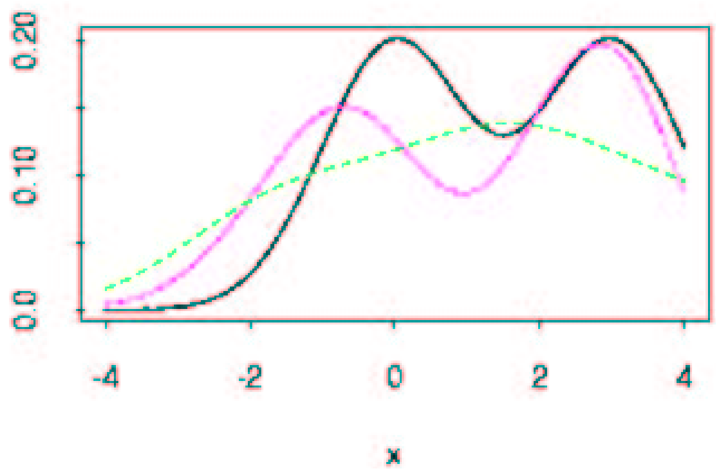
$Y_j = X_j + U_j$ has a density

$$
\begin{aligned}
g_j(y) &= \frac{F(y+h_j) - F(y-h_j)}{2h_j}, \quad F = F_X \\
\Longrightarrow \quad F(x) &= 2 \sum_{k=\text{odd}} \frac{1}{n} \sum_{j=1}^{n} h_j g_j(x - kh_j) \\
\Longrightarrow \quad \hat{F}_-(x) &= 2 \sum_{k=\text{odd}} \frac{1}{n} \sum_{i=1}^{n} h_i w_b(x - kh_i - Y_i) \\
\hat{f}_-(x) &= 2 \sum_{k=\text{odd}} \frac{1}{n} \sum_{i=1}^{n} h_i w_b'(x - kh_i - Y_i)
\end{aligned}
$$

where

$$w_b(x) = \frac{1}{b} w(\frac{x}{b}).$$

Comparisons of Estimators



Notes: $\hat{f}_{-}(x)$ is much better than the $\hat{f}_{naive}(x)$, where the

true density is $f(x)$ in black.

Improvements: look at another side

Similarly

$$
\begin{aligned}
g_j(y) &= \frac{1 - F(y - h_j) - (1 - F(y + h_j))}{2h_j} \\
\Longrightarrow \quad F(x) &= 1 - 2 \sum_{k=\text{odd}} \frac{1}{n} \sum_{j=1}^{n} h_j g_j(x + kh_j) \\
\Longrightarrow \quad \hat{F}_+(x) &= 1 - 2 \sum_{k=\text{odd}} \frac{1}{n} \sum_{i=1}^{n} h_i w_b(x + kh_i - Y_i) \\
\hat{f}_+(x) &= -2 \sum_{k=\text{odd}} \frac{1}{n} \sum_{i=1}^{n} h_i w_b'(x + kh_i - Y_i),
\end{aligned}
$$

Proposition 1. Let $H = \sum h_j$. Then under modest conditions on b, h_j and f,
$\hat{F}^+, \hat{F}^-$ are asymptotically independent, and

$$
\begin{aligned}
\hat{F}_-(x) &\sim N\left(F(x), F(x)\frac{2H\|w\|^2}{n^2 b}\right) \\
\hat{F}_+(x) &\sim N\left(F(x), (1 - F(x))\frac{2H\|w\|^2}{n^2 b}\right)
\end{aligned}
$$

Idea: use the Combined Estimator:

$$\tilde{F}(x) = [1 - p(x)]\hat{F}_-(x) + p(x)\hat{F}_+(x),$$

where p is a distribution function.

Choices of p

Adhoc: $p = e^x/(1+e^x)$.

$$\tilde{F}(x) = [1 - \frac{e^x}{1+e^x}]\hat{F}_-(x) + \frac{e^x}{1+e^x}\hat{F}_+(x)$$

MVUE: $p = F$

$$\hat{F}(x) = [1 - \hat{F}(x)]\hat{F}_-(x) + \hat{F}(x)\hat{F}_+(x)$$
$$\hat{F}(x) = \frac{\hat{F}_-(x)}{\hat{F}_-(x) + 1 - \hat{F}_+(x)},$$

"Optimal" Estimates of Densities

$$\begin{aligned}\tilde{f}(x) &= [1 - p(x)]\hat{f}_-(x) + p(x)\hat{f}_+(x) \\ &\quad +p'(x)[\hat{F}_+(x) - \hat{F}_+(x)]\end{aligned}$$

$$\hat{f}(x) = \frac{[1 - \hat{F}(x)]\hat{f}_-(x) + \hat{F}(x)\hat{f}_+(x)}{\hat{F}_-(x) + 1 - \hat{F}_+(x)}$$

MSE

Propostion 2. Under the same conditions of Proposition 1, let $\bar{h} = H/n = O(1)$, then

$$
\begin{aligned}
E\hat{F}_+(x) &= E\hat{F}_-(x) = w_b * F(x), \\
E\hat{f}_+(x) &= E\hat{f}_-(x) = w_b * f(x), \\
\text{MSE}(\hat{F}) &= w_2^2 \frac{b^4}{4} f'^2(x) + \frac{2\bar{h}}{nb} \|w\|^2 F(x)[1-F(x)] \\
&\quad + o(b^4) + o(\frac{1}{nb}), \\
\text{MSE}(\hat{f}) &= w_2^2 \frac{b^4}{4} f''^2(x) + \frac{2\bar{h}}{nb^3} \|w'\|^2 F(x)[1-F(x)] \\
&\quad + o(b^4) + o(\frac{1}{nb^3}).
\end{aligned}
$$

Optimal b

The optimal window widths minimizing MISE of $\hat{F}$ and $\hat{f}$ are:

$$
\begin{aligned}
b_F &= n^{-1/5} (\frac{2\bar{h}\|w\|^2 c}{w_2^2 \|f'\|^2})^{1/5}, \\
b_f &= n^{-1/7} (\frac{6\bar{h}\|w'\|^2 c}{w_2^2 \|f''\|^2})^{1/7}
\end{aligned}
$$

resepctively, where $c = c(F) = \int F(x)(1-F(x))dx$.

Rates	$b = O(n^{-1/5})$	$b = O(n^{-1/5})$
$\hat{F}$	$n^{-2/5}$	$n^{-1/7}$
$\hat{f}$	$n^{-1/5}$	$n^{-1/7}$

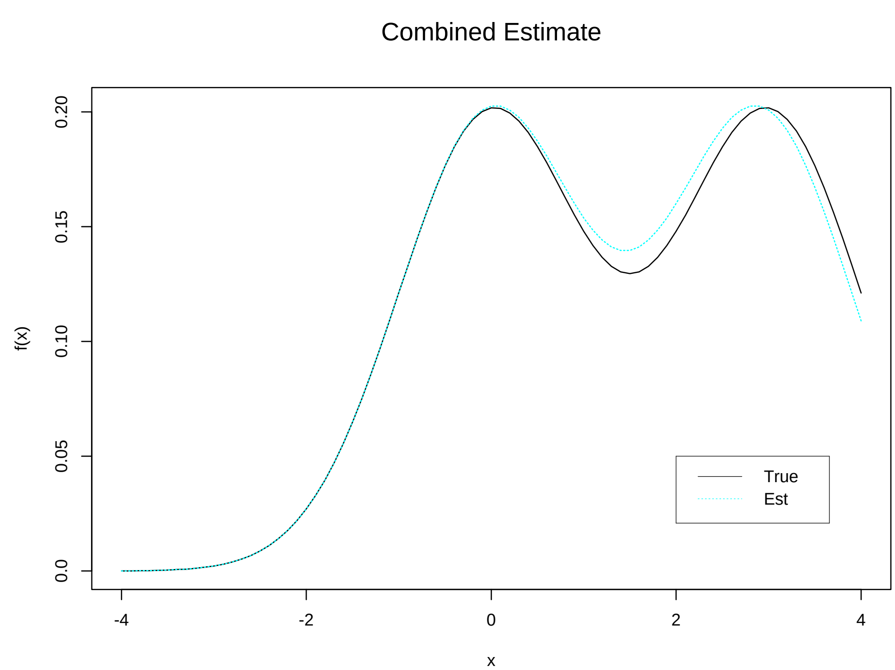
Practical choices of b

We suggest 2 ways of computing b_f. The 1st is to estimate f'' and $c(F)$ based on a pilot kernel estimate of f with a larger window width, say $b = n^{-1/7+1/5} b_w$. The justification of this method can be based on that of bootstrap (Delaigle A. and Gijbels I, (2001)). The 2nd can be obtained by tracking similar arguments to that of Silverman's rule of thumb:

$$\hat{b}_F = n^{-1/5} 1.0324 (R^4 \bar{h})^{1/5}, \quad \hat{b}_f = n^{-1/7} 0.9763 (R^6 \bar{h})^{1/7}.$$

where

$$R = \min\{[\mathrm{Var}(Y) - (\bar{h})^2/3]^{1/2}, \frac{IQR(Y)}{1.34}\}.$$



Solution 2
Smooth out the singularity points

Naive deconvolution

→ Crude singularity adjusted deconvolution

→ Cosine bell adjusted deconvolution

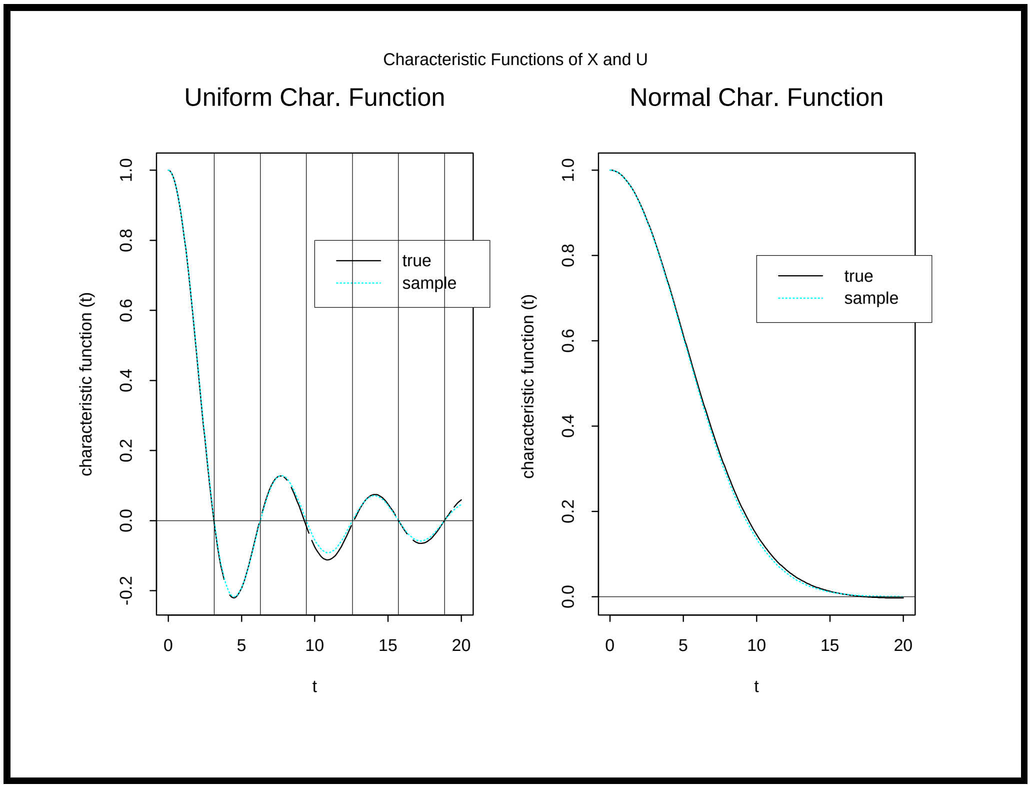
→ Shannon singularity correction!

Two Examples using Solution 2

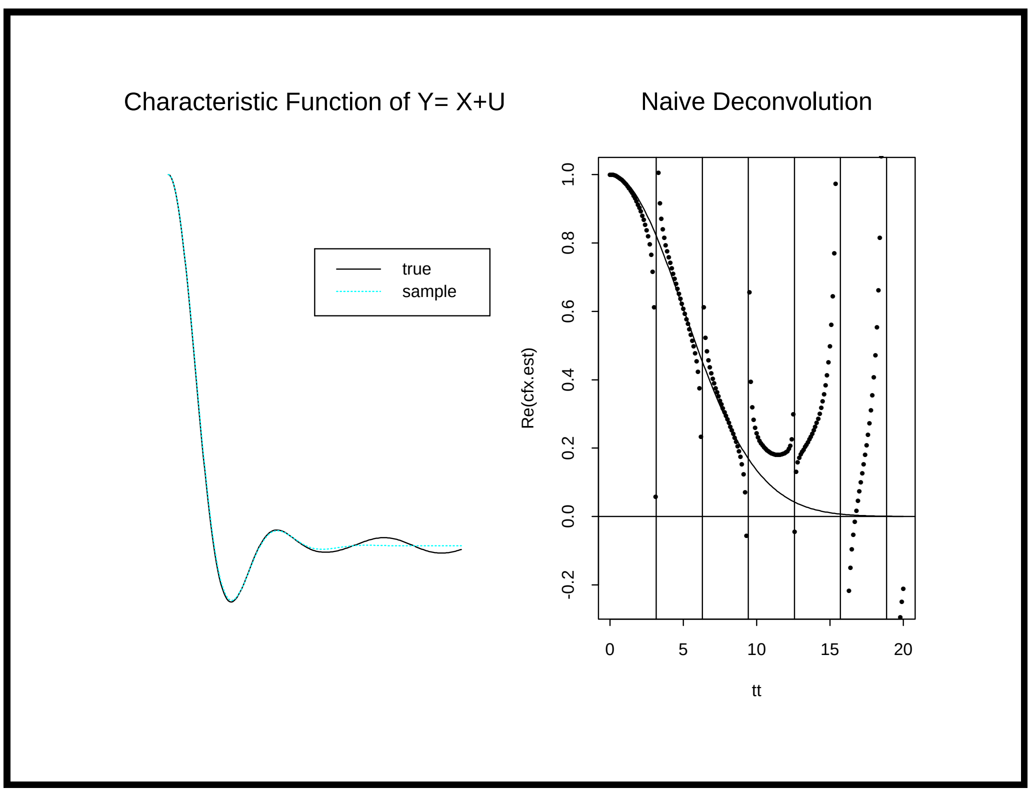
- Example 1:

$X \sim N(0, 0.2^2)$, $U \sim \mathcal{U}(0,1)$

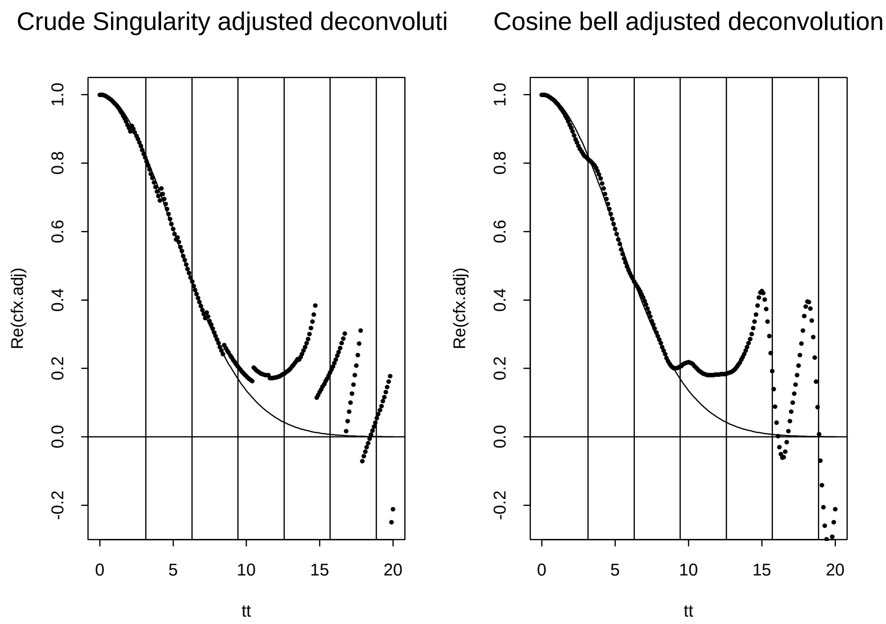
to motivate the final solution


Characteristic Functions of X and U
Uniform Char. Function
Normal Char. Function
characteristic function (t)
true
sample
t
0
5
10
15
20
-0.2
0.0
0.2
0.4
0.6
0.8
1.0

25


Characteristic Function of Y= X+U
Naive Deconvolution
true
sample
Re(cfx.est)
tt
0
5
10
15
20
-0.2
0.0
0.2
0.4
0.6
0.8
1.0

26

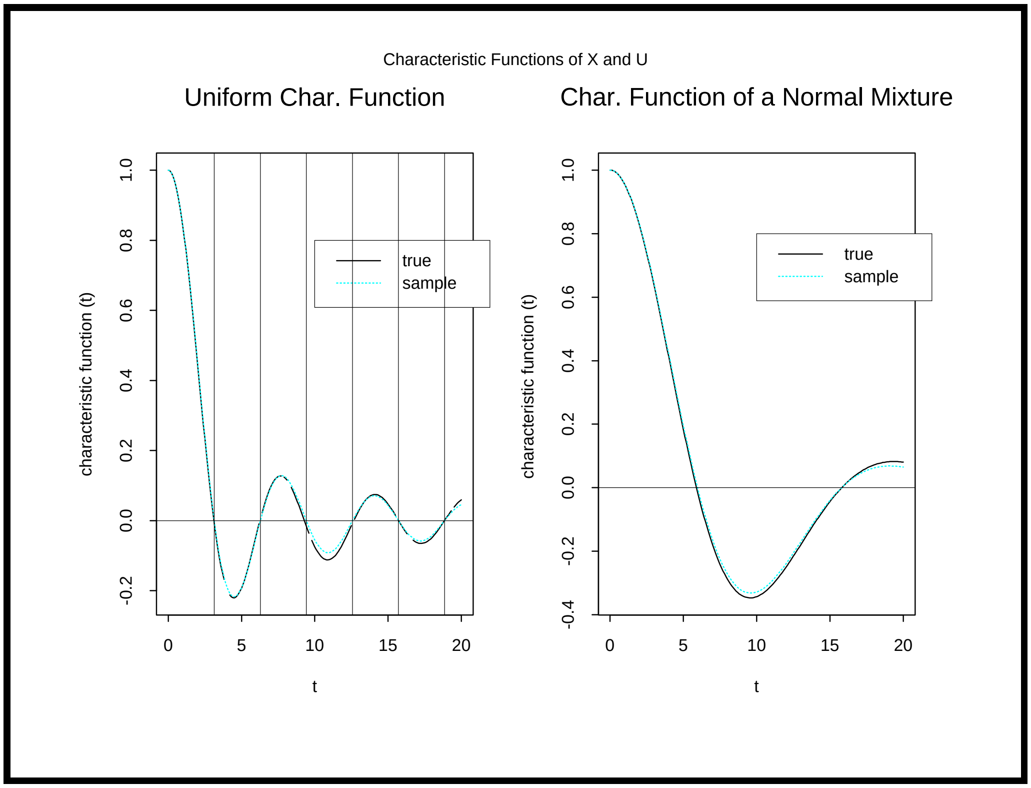


Example 2

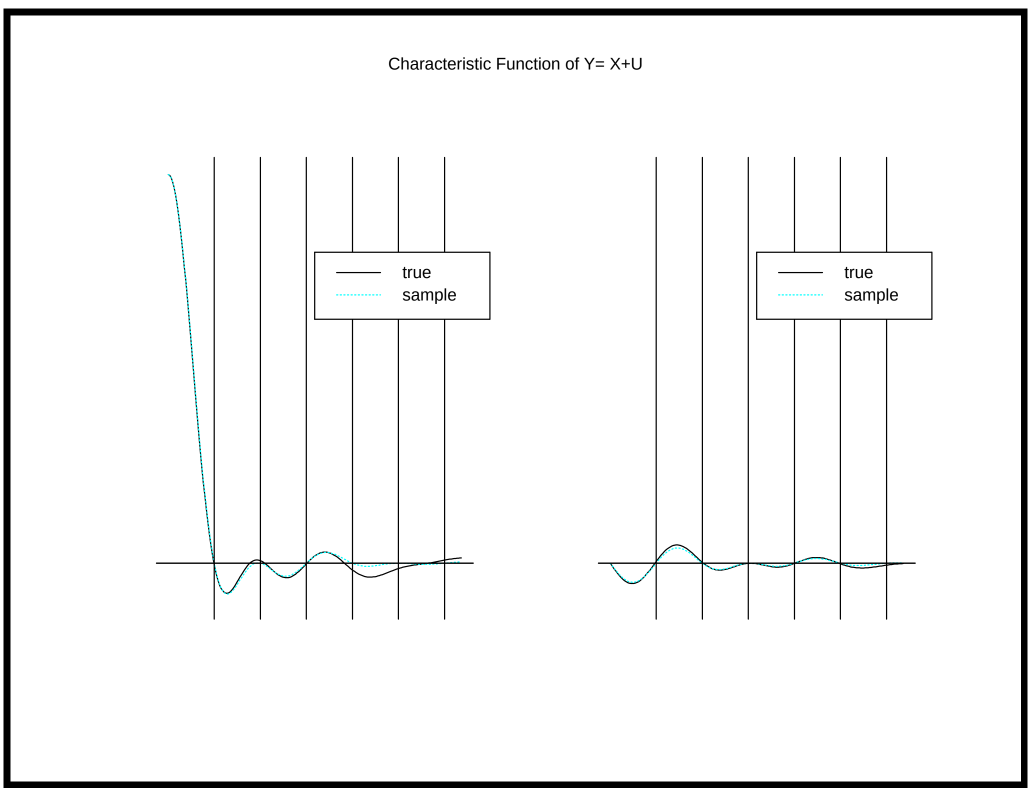
- Example 2:

$X \sim 0.5 \cdot N(0.2, 0.2^2) + 0.5 \cdot N(-0.3, 0.1^2)$, $U \sim \mathcal{U}(0,1)$

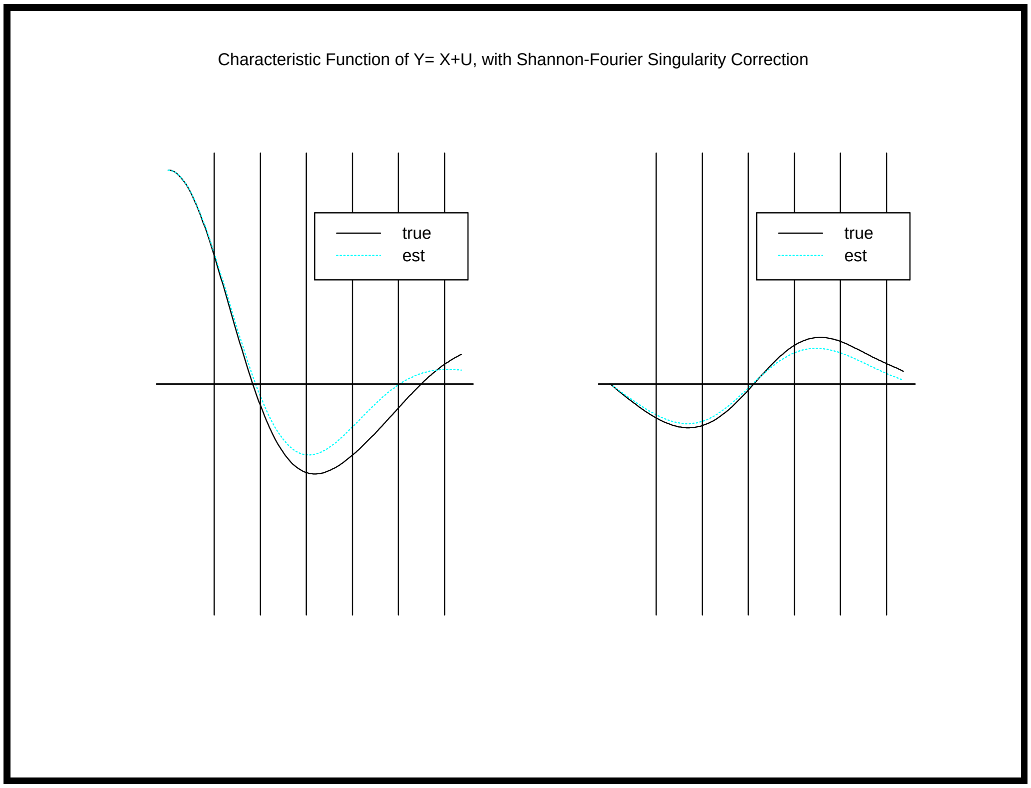
to check how the final estimate work


Characteristic Functions of X and U
Uniform Char. Function
Char. Function of a Normal Mixture
characteristic function (t)
t
true
sample
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
0
5
10
15
20

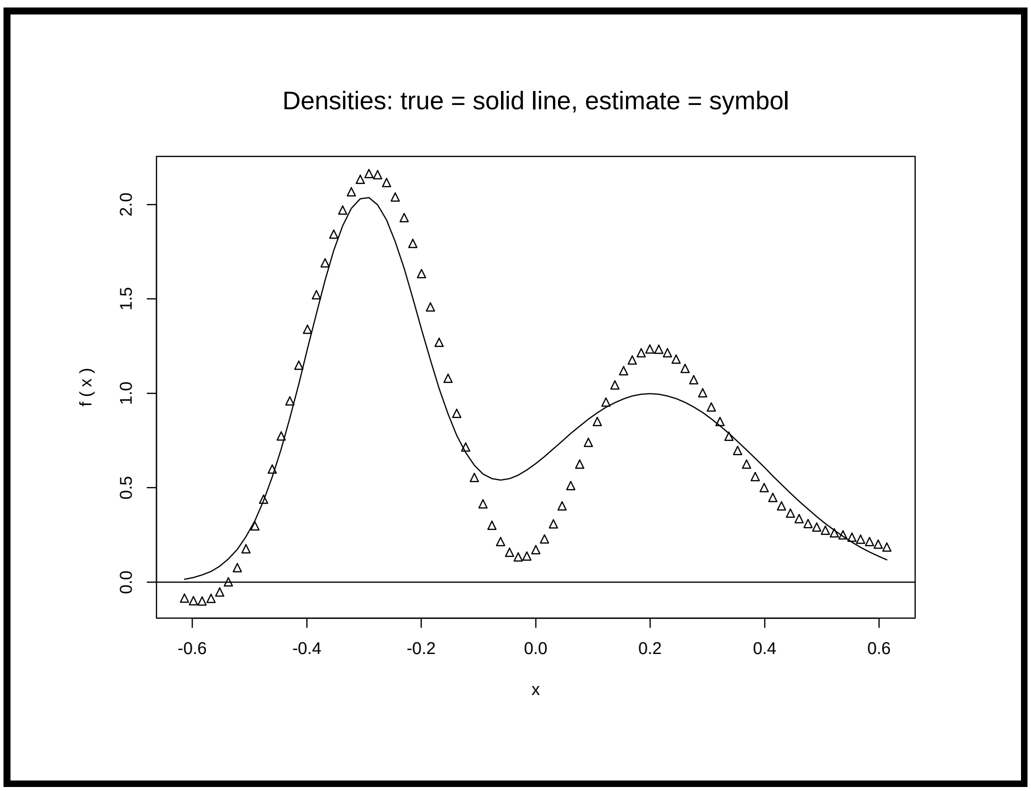
29


Characteristic Function of Y= X+U
true
sample

30


Characteristic Function of Y= X+U, with Shannon-Fourier Singularity Correction
true
est
true
est

31


Densities: true = solid line, estimate = symbol
f (x)
0.0
0.5
1.0
1.5
2.0
-0.6
-0.4
-0.2
0.0
0.2
0.4
0.6
x

32

Discussion and Remarks

1. Connection to Regression:

$$\varphi_{Y,n}(t) = \varphi(t) + e(t) = \varphi_X(t)\varphi_U(t) + e(t) \qquad (2)$$

   where the multiplier function $\varphi_U(t)$ is known and the covariance of $e(t)$ is easy to estimate. It has a connection to image analysis - see p6:

2. Splines - two folded:
   - Model $\varphi_X(t)$ in (2), $\varphi_X(t) = \beta^T \mathbf{s}(t)$.
   - Smooth $\hat{\varphi}_Y/\varphi_U$ - see p9 and jhf.ps.

3. Generalization to non-homogeneous errors:

$$\hat{f}_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{n}\sum_j \frac{e^{itY_j}}{\varphi_{U_j}(t)} \; e^{-itx}\,dt$$

4. Comparisons of modified deconvolution and non-deconvolution estimates.

5. In non-homogeneous case, there is one data point for each error distribution. Are the final estimates (plots) much different under the normal and uniform error models (with the uniform and normal variances matched in some ways)?

6. Application in deblurring optical images that have been subjected to uniform motion over a finite interval of time, and other errors.